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WAVE SUPPRESSION UNDER THE ACTION OF A VISCOUS FILM ON THE SURFACE OF AN IDEAL FLUID*

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We use below the linearized Euler's equations for an ideal fluid occupying the lower half-space, and the Navier-Stokes equations for a layer of viscous fluid of vanishingly small thickness lying above it (viscous film), to obtain the solution of the problem of gravitation-capillary waves on the surface of an ideal fluid produced by the action of surface tension. The conditions under which the viscous film seriously reduces the wave amplitude are found.

According to existing theorems /1-3/, suppression of the surface film waves occurs either as a result of a reduction in the coefficient of friction at the phase interface, so that the energy transmitted to the fluid from outside is reduced, or else as a result of energy dissipation on deformation of the viscous film, or as a result of the appearance on the fluid surface of extra friction forces, dependent on the velocity, which lead to the vortical part of the motion being intensified, so that the energy dissipation of the lower fluid is increased.

We discover in the present paper a further effect of film suppression which is independent of energy dissipation either in the film itself or in the fluid. The fact is that the tangential stresses on the film outside boundary are transformed into normal stresses, which are transmitted to the fluid below. Under certain conditions these extra normal stresses cause waves on the fluid boundary which have a phase shift relative to the waves caused by the normal stress. The wave interference can cause serious attenuation of the total wave.

1. The plane problem of the wave motion of a two-layer flow (without velocity shift), consisting of a layer of ideal fluid of infinite depth and a viscous layer of constant thickness h, reduces to the boundary value problem /4, 5/

$$
\begin{gathered}
c\frac{\partial \mathbf{u}_1}{\partial x} = -\frac{1}{\rho_1}\nabla P_1 + \nu\Delta\mathbf{u}_1, \quad c\frac{\partial \mathbf{u}_2}{\partial x} = -\frac{1}{\rho_2}\nabla P_2 \\
\nabla\cdot\mathbf{U}_1 = 0, \ \nabla\cdot\mathbf{u}_2 = 0 \\
P_1 = p_1 + \rho_1 gz, \ P_2 = p_2 + (\rho_2-\rho_1)gh + \rho_2 gz \\
-P_1 + \rho_1 g\zeta_1 - \alpha_1\frac{\partial^2\zeta_1}{\partial x^2} + 2\rho_1\nu\frac{\partial u_{12}}{\partial z} = -P_* \\
\rho_1\nu\left(\frac{\partial u_{11}}{\partial z} + \frac{\partial u_{12}}{\partial x}\right) = \tau_*, \quad c\frac{\partial\zeta_1}{\partial x} = u_{12}, \quad z = 0 \\
-P_2 + (\rho_2-\rho_1)g\zeta_2 - \alpha_2\frac{\partial^2\zeta_2}{\partial x^2} = -P_1 + 2\rho_1\nu\frac{\partial u_{12}}{\partial z} \\
\frac{\partial u_{11}}{\partial z} + \frac{\partial u_{12}}{\partial x} = 0, \quad c\frac{\partial\zeta_2}{\partial x} = u_{22} = u_{12}, \quad z = -h \\
\Psi_1 \to 0, \ z\to -\infty; \ \Psi_2 \to 0, \ |x|\to\infty \\
\left(\Psi_1 = \{P_2, \mathbf{u}_2\}\right. \\
\left.\Psi_1 = \left\{P_j, \mathbf{u}_j, \frac{\partial\mathbf{u}_j}{\partial x}, \zeta_j, P_*, \tau_*\right\}, \ j = 1, 2\right)
\end{gathered}
\tag{1.1}
$$

Here, c is the fluid flow velocity, α_1, α_2 are the surface tension at the free surface and the interface, ρ_1, ρ_2 are the densities, ν is the kinematic viscosity of the upper layer, g is the acceleration due to gravity, $\mathbf{u}_1 = \{u_{11}, u_{12}\}$, $\mathbf{u}_2 = \{u_{21}, u_{22}\}$ are the velocity vectors, and p_1, p_2 are the hydrodynamic pressures (subscript 1 refers to the upper, and 2 to the lower, fluid). The origin is on the undisturbed surface of the upper layer, and the z axis is vertically upwards.

Let the active load be

$$
P_*(x) = \frac{4Q}{\pi}\frac{a_1}{a_1^2+x^2}, \quad \tau_*(x) = \frac{T}{\pi}\frac{xf}{(f^2+x^2)^2}
$$

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We introduce the dimensionless quantities

$$y=\frac{x}{f},\quad a=\frac{a_1}{f},\quad \varepsilon=\frac{h}{f},\quad \eta_2=\frac{\zeta_2}{f},\quad \gamma=\frac{\rho_1}{\rho_2}$$
$$F=\frac{c^2}{gf},\quad R=\frac{cf}{\nu},\quad \beta_1=\frac{\alpha_1}{\rho_1 g f^2},\quad \beta_2=\frac{\alpha_2}{\rho_2 g f^2}$$
$$\lambda=\frac{Q}{\rho_2 g f},\quad \delta=\frac{T}{\rho_2 g f^3},\quad \beta=\gamma\beta_1+\beta_2,\quad \beta_3=\beta_2-\beta_1(\gamma-1)$$

where f is a parameter characterizing the size of the zone of application of the tangential stresses. Then, applying a Fourier integral transformation to (1.1) with respect to x, we find the rise of the interface in the integral form

$$\eta_2=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\Delta_1}{v\Delta}\exp(-ivy)\,dv$$
$$\Delta=i\,|v|^3R^{-1}F^{-2}\{FR^{-2}B_1(4S_1C_2\,|v|\,b-C_1S_2B_4\times\operatorname{sign}v)-$$
$$S_1S_2[\,|v|\,B_1B_2+\gamma v^2B_2^2-\gamma F^2R^{-4}(16v^2b^6+B_3^4)]+8\gamma F^2R^{-4}\times v^3bB_4(1-$$
$$C_1C_2)\}$$
$$\Delta_1=F^{-2}[2iS_1v^3b(2P_{**}v^2FR^{-1}-\tau_{**}B_2)+2\tau_{**}v^3bFR^{-1}B_3(C_2-C_1)+$$
$$iS_2v^2B_3(P_{**}F\times R^{-1}B_3+\tau_{**}B_2)],$$
$$B_1=Fv^2-\beta\,|v|^3-|v|,\ B_2=1+\beta_1v^2,\ B_4=(2v-iR)^2,\ b^4=v^2B_4$$
$$B_3=v^2+b^2,\ S_1=\operatorname{sh}\varepsilon v,\ S_2=\operatorname{sh}\varepsilon b,\ C_1=\operatorname{ch}\varepsilon v,\ C_2=\operatorname{ch}\varepsilon b$$

where $P_{**},\ \tau_{**}$ are the Fourier transformants of $P_*,\ \tau_*$.

2. Our further analysis is for the case when the upper viscous layer is vanishingly thin. Retaining the first two terms of the expansion of Δ and Δ_1 in powers of ε, we obtain

$$\eta_2=\eta_{20}+\varepsilon\eta_{21}+O(\varepsilon^2) \tag{2.1}$$
$$\eta_{20}=-\frac{\lambda}{2\pi}\int\limits_{-\infty}^{\infty}\varkappa\,dv-\frac{i\delta R}{2\pi F}\int\limits_{-\infty}^{\infty}\frac{\beta_1v^2+1}{4v-iR}\varkappa\,dv \tag{2.2}$$
$$\varkappa=\exp(-a\,|v|-ivy)(\beta v^2-F\,|v|+1+\varepsilon f_2)^{-1}$$
$$f_2=iRF^{-1}(\beta_1\beta_2v^4-F\beta_1\,|v|^3+\beta_3v^2-F\,|v|-\gamma+1)(4v-iR)^{-1}+$$
$$\gamma F^2R^{-1}v^2$$

While the second term η_{21} is not quoted, we take it into account in the numerical analysis. Notice that, with $\varepsilon=0$, the same result is obtained from (2.1) as when solving the problem on the motion of an ideal fluid covered by a thin viscous film when the equations of the theory of long waves of a viscous fluid are used.

The first integral in (2.2) corresponds to the wave resulting from the action of normal stresses on the surface of the ideal fluid (this result, with $\varepsilon=0$, was obtained in /2/). The second integral corresponds to the wave due to the action of the extra normal stresses transmitted to the lower fluid from the tangential stresses acting on the film surface. The amplitude of this wave depends on the intensity τ_* of the tangential stresses. Under certain conditions imposed on the physical parameters, the interference of these waves causes considerable reduction of the total wave amplitude.

To illustrate this, let us evaluate the integral in (2.2). The integration is performed in four separate cases: $F^2\gtrless 4\beta$ and $y\gtrless 0$. The technique of evaluation is described in /2/, and the tabulated integrals are taken from /6/. We have

$$\eta_{20}=\exp(-\varepsilon yv_*(v))(\operatorname{sign}y)[-\lambda(F^2-4\beta)^{-1/2}\times\exp(-va)\sin yv+ \tag{2.3}$$
$$\delta\Lambda(1+\beta_1v^2)\exp(-v)(4v\cos yv+R\sin yv)]+\eta_+$$
$$F^2-4\beta\geqslant\delta_*>0,\quad v=\begin{cases}v_2, & y>0\\ v_1, & y<0\end{cases}$$
$$v_{1,2}=(F\pm\sqrt{F^2-4\beta})/(2\beta),\ v_*(v)=4v(-\beta_1\times\beta_2v^4+F\beta_1v^3-\beta_3v^2+$$
$$Fv+\gamma-1)\Lambda,\ \Lambda=[RF(1+16v^2R^{-2})\sqrt{F^2-4\beta}]^{-1}$$

$$\eta_+=\operatorname{Re}\left\{\frac{\lambda}{\pi\sqrt{F^2-4\beta}}[E((a+iy)v_1)-E((a+iy)v_2)]+\right. \tag{2.4}$$
$$\frac{i\delta R(\beta_1R^2-16)}{4\pi F(\beta R^2+4iRF-16)}E\left(\frac{R}{4}(y-i)\right)-\frac{i\delta R}{\pi F\sqrt{F^2-4\beta}}\times$$
$$\sum_{j=1}^{2}(-1)^j\frac{\beta_1v_j^2+1}{4v_j-iR}E((1+iy)v_j)$$
$$E(q)=\exp(-q)\operatorname{Ei}(q)$$

(Ei (x) is the integral exponential function).

Similarly, we find the solution when $4\beta - F^2 \geqslant \delta_* > 0$

$$\eta_{20} = \frac{\delta R}{s\gamma\beta(64F^2 + 2s - R\beta)}\left[A \sin\frac{Fy - s}{2\beta} - B\cos\frac{Fy - s}{2\beta}\right] \times \exp\left(-\frac{sy + F}{2\beta}\right) - \frac{\lambda}{s}\exp\left(-\frac{sy + Fa}{2\beta}\right)\cos\frac{Fy - sa}{2\beta} + \eta_+ \quad (2.5)$$

$$A = s(\beta + \gamma\beta_1)(R\beta - 2s) - 8M$$

$$B = F^{-1}[(R\beta - 2s)M + 8F^2 s(\beta + \gamma\beta_1)]$$

$$s = \operatorname{sign} y\sqrt{4\beta - F^2}, \quad M = 2F^2\gamma\beta_1 - 4\gamma\beta(\beta_1 - \beta)$$

The amount of deformation of the interface falls exponentially as s or $|y|$ increases.

Consider the wave motion of the fluid remote from the domain of disturbance. Then, (2.3)-(2.5) remain true, where we use for η_+ the asymptotic expansion of the integral exponential. Retaining the first term, we obtain

$$\eta_+ \sim -\frac{\lambda a}{\pi(a^2 + y^2)} - \frac{\delta}{\pi F(1 + y^2)}\{\gamma^{-1} - R\beta^{-2}[4F\beta y(\beta_1 R^2 - 16) + R^3\beta(\beta + \beta_1) + 8RF^2 + 16R\beta_1 + R^{-1}(16 - \beta_1 R^2)(4RFy - R^2 + 16)] \times (\beta^2 R^4 + 16R^2F^2 + 256)^{-1}\}$$

It follows from our results that, as the viscous film ν tends to zero, the contribution of the tangential stresses to the interface deformation remains finite. With $F^2 - 4\beta \geqslant \delta_* > 0$ we have

$$\eta_{20} \sim \begin{cases} \eta(v_2), & y > 0 \\ \eta(v_1), & y < 0 \end{cases} \quad (2.6)$$

$$\eta(v) = [\delta F^{-1}(1 + \beta_1 v^2)\exp(-v) - \lambda\exp(-va)] \times \frac{\sin yv}{\sqrt{F^2 - 4\beta}} - \frac{\lambda a}{\pi(a^2 + y^2)} - \frac{\delta(\beta_2 - \gamma\beta)}{\pi\gamma\beta F(1 + y^2)}$$

It should be noted that the result (2.6) is independent of the small film thickness h and of the small viscosity ν. Thus a weakly viscous thin film greatly reduces the wave amplitude and deforms the interface. If $F^2 < 4\beta$, system of waves arises, localized in the domain of action of the external load, whose amplitude falls exponentially with distance. A similar result was obtained in /1/ for a fluid without a film.

The dependence of the wavelength L on the parameter $\psi = F/\sqrt{\beta}$ is shown in Fig.1. With $\psi > 2$ (which corresponds to $F^2 > 4\beta$) the upper and lower branches describe the length variation of the gravitational and capillary waves. It can be seen that, as ψ increases, the length of the gravitational wave increases and the length of the capillary wave decreases. For $\psi < 2$ the lengths of both waves become the same and increase as ψ decreases /2/.

Minimization methods were used to find the physical parameters for which the total wave amplitude is saml1 compared with the amplitude of the wave arising under the action of normal stresses only. Below we give an example which shows clearly how wave suppression occurs when external tangential stresses are introduced: $A_{10} = 1.03\cdot10^{-3}$, $A_{20} = 1.15$, $\delta = 0.93$, $\lambda = 1$, $a = 1$, $\beta_1 = 0.489$, $\beta_2 = 0.275$, $\gamma = 0.551$, $R = 5000$, $F = 1.5$, $A_{11} = 1.31$, $\varepsilon_* = 7.88\cdot10^{-5}$.

Here, A_{10} is the total wave amplitude, A_{20} is the wave amplitude under the action of normal external stresses only, A_{11} is the first correction to the total wave amplitude A_1 when it is expanded in powers of the small parameter ε ($A_1 = A_{10} + \varepsilon A_{11} + O(\varepsilon^2)$, $A_{11} = \max_y|\eta_{21}|$), ε_* is the maximum tolerable upper bound of ε. In the calculations the choice of ε_* was based on the requirement that $|\varepsilon_* A_{11}/A_{10}| = 10^{-2}$. It can be seen that, when tangential stresses on the film surface are taken into account, we get a decrease of three orders in the wave amplitude. We calculate A_{10} and A_{20} from (2.4) with $\varepsilon = 0$.

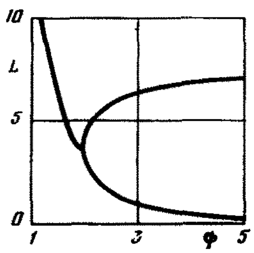

Fig.1

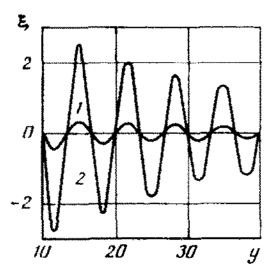

Fig.2

In Fig.2 we show the wave suppression by a viscous film (only the wave part of the interface deformation is shown): *1* is the total wave arising under the action on the film surface of normal and tangential stresses, *2* is the wave arising under the action of normal stresses only, $\eta_{20} = 0.01\xi$. The parameter values are $\delta = 0.93$; $\lambda = 4.99$, $a = 5$, $\beta_1 = 12.3$, $\beta_2 = 0.0025$, $\gamma = 0.0037$, $R = 500$, $F = 1.09$. In this case the thin film reduces the wave amplitude by one another.

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A HYDRODYNAMIC MODEL OF AN INTERCEPTING VACUUM DRAINAGE IN A DESCENDING FLOW OF GROUND WATER*

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A boundary value problem is formulated describing two-dimensional steady filtration in a layer of soil of infinite capacity, towards a horizontal vacuum drainage which captures partly or wholly, at some specified rate of drainage, the ground water filtering downwards from the soil surface. A solution of the problem is constructed using conformal mapping and the solution contains two unknown mapping parameters. A system of equations is derived for the latter parameters, and their unique solvability is established analytically. At the same time, a restriction on the filtration capacity of the drainage is revealed, corresponding to the critical mode (complete interception of the flow). A computer program is written for the algorithm used to compute, for the given parameters of the medium, the flow characteristics in the critical mode, and at some value of the drainage output chosen arbitrarily from the interval of admissible values. Numerical examples are given.

Horizontal vacuum drains are installed in the irrigated soil of infinite capacity. Their purpose is to intercept the irrigation water seeping through the root system from the surface, flooded with a thin layer of water. We will assume that every stream associated with the action of one or another drain is symmetrical about a vertical line passing through the centre of the drain. The side boundaries of the streams are free, and atmospheric pressure is maintained along them. For this reason the flows towards separate drains are themselves separate, so that we can speak of a system of drains in the model in question only in a conventional manner.

The right half of the zone of action of one of the drains represented by a point drain B, (sink), is shown schematically in the $z = x + iy$ -variable plane as a region of flow in Fig.1a for the case when a proportion Q of the total (within the shaded region) inflow Q_s of surface water is intercepted by the drain, and the remainder Q_0 flows down to infinity. The corresponding domain of the complex potential $\omega = \varphi + i\psi$ is shown in Fig.1b. The investigation is carried

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